LES Group Extensions

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It is well understood that a short exact sequence defines a group extension problem

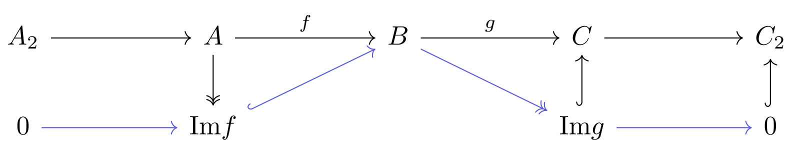
$$0 \to A \to B \xrightarrow{g} C \to 0$$

as by the first isomorphism theorem applied to g we see that $C \cong B/A$, that is B is "A extended by C" or vice versa. We would like to write that $B \cong A \times B$ but this is only the case when the sequence splits and so there is a difference between "extending" in general and products.

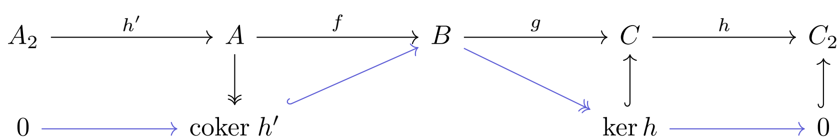
Now if on the other hand we had a long exact sequence of groups

$$\cdots \to A_2 \to A \to B \to C \to C_2 \to \cdots$$

then what extension problem does it define (if any)? Well at any point in the sequence we can restrict to the image of the maps to get something short exact



where the purple arrows are exact because they come from the LES. Or we can use the exactness of the LES to rewrite these groups in other ways



because $A/\ker f = A/\mathrm{Im}\ h' = \mathrm{coker}\ h'$. Thus a LES describes an extension problem at ever place of the sequence.